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Finsler metrics with positive constant flag curvature

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Dedicated to Professor Karsten Grove on the occasion of his 60th birthday

Abstract. We showed that any reversible Finsler metric with positive constant flag curvature must be Riemannian.

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Introduction. One of the fundamental problems in Finsler geometry is to describe the Finsler metrics of positive constant flag curvature. In this case, Shen [13] asserts that the Finsler manifold must be a diffeomorphic sphere, provided that it is simply connected. Bryant [5] has constructed a family of non-reversible Finsler metrics of positive constant flag curvature on spheres which are projectively flat. Bejancu and Farran [4] have proved that a Finsler manifold is of positive constant flag curvature one if and only if the unit horizontal Reeb vector field is a Killing vector field on the unit tangent bundle. However, the geometric structure of such metric still remains mysterious (see [8, 9]).

Kim and Yim [10] showed that if reversible Finsler metrics with positive constant flag curvature have vanishing mean tangent curvature, then it is Riemannian. Likewise, it has been recently remarked by Bryant [6] that the fundamental and deeper result of LeBrun and Mason, in [11], implies a reversible Finsler metric on the two-dimensional sphere with positive constant flag curvature is Riemannian. Our main theorem below improves the previous result into manifolds with dimension greater than two.

Theorem. *Any reversible Finsler metric with positive constant flag curvature must be Riemannian.*

Bao and Shen [3] also constructed a family of Randers metrics with positive constant flag curvature which are not Riemannian. These Finsler metrics are non-reversible, and hence they do not contradict the main theorem.

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1. Preliminaries. In this section, we shall recall some well-known facts about Finsler geometry. See [2], for more details. Let M be an n-dimensional smooth manifold and TM denote its tangent bundle. A *Finsler structure* on a manifold M is a map $F : TM \to [0, \infty)$ which has the following properties

(a) F is smooth on $\widetilde{TM} := TM \setminus \{0\}$;
(b) $F(ty) = tF(y)$, for all $t > 0,\ y \in T_xM$;
(c) F^2 is strongly convex, i.e.,

$$g_{ij}(x,y) := \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}(x,y)$$

is positive definite for all $(x, y) \in \widetilde{TM}$.

A manifold M endowed with a Finsler structure will be called a Finsler manifold. Note that we never require smoothness at the zero section. Requiring that $F(x,\cdot)$ is smooth on all of T_xM implies that $F(x,\cdot)$ is a Hilbert norm (see [14]). If condition (b) holds for all t we will say that the Finsler structure F is reversible. A Minkowski space is a finite dimensional real vector space that has a Finsler metric independent of x, $F(x,y) = F(y)$. Let F_x denote the restriction of F onto T_xM. When F is Riemannian, (T_xM, F_x) are all isometric to the Euclidean space $\mathbb{R}^n$. For a general Finsler metric F, however, the Minkowski space (T_xM, F_x) may not be isometric to each other.

The Minkowski metric F_x induces a Riemannian metric $\hat{g}_x$ on $T_xM \setminus \{0\}$ by

$$\hat{g}_x(u,v) := g(x,y)(u,v), u, v \in T_yT_xM = T_xM.$$

Let $\{y^i\}_{i=1}^n$ be a global coordinate system in T_xM associated with a basis $\{b_i\}_{i=1}^n$. The Riemannian volume form of $\hat{g}_x$ is given by

$$\begin{aligned} d\hat{g}_x &= \sqrt{\det\big(g_{ij}(x,y)\big)}\ dy^1 \wedge \cdots \wedge dy^n \\ &= \sqrt{\det\big(g_{ij}(x,y)\big)}\ dy, \end{aligned}$$

where $g_{ij}(x,y) = g(x,y)(b_i, b_j)$. For $x \in M$, let $S_xM := \{y \in T_xM : F(x,y) = 1\}$ denote the unit tangent sphere at x, and $D_xM := \{y \in T_xM : F(x,y) \le 1\}$ the unit tangent disk at x. Let $\dot{g}_x$ denote the induced Riemannian metric on S_xM.

Denote by $\mathrm{vol}_{\hat{g}_x}(D_xM)$ and $\mathrm{vol}_{\dot{g}_x}(S_xM)$ the Riemannian volume of $(D_xM, \hat{g}_x)$ and $(S_xM, \dot{g}_x)$, respectively. By Gauss' Lemma, we have $d\hat{g}_x(ty) = t^{n-1}d\dot{g}_x(y)$ and hence

$$\begin{aligned}\mathrm{vol}_{\hat{g}_x}(D_xM) &:= \int\limits_{D_xM} 1 \; d\hat{g}_x \\ &= \int\limits_0^1 t^{n-1}\left\{ \int\limits_{S_xM} 1 \; d\dot{g}_x \right\} dt = \frac{1}{n}\,\mathrm{vol}_{\dot{g}_x}(S_xM).\end{aligned}$$

Lemma 1.1. *Let (M, F) be an n-dimensional reversible Finsler manifold. Then for all $x \in M$,*

$$\mathrm{vol}_{\hat{g}_x}(D_xM) \leq \beta(n), \;\; \mathrm{vol}_{\dot{g}_x}(S_xM) \leq \alpha(n-1),$$

where $\beta(n)$ is the volume of the unit n-ball $\mathbb{B}^n$ in Euclidean space $\mathbb{R}^n$ and $\alpha(n-1)$ is that of the unit $(n-1)$-sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$, and equality holds if and only if (T_xM, F_x) is Euclidean.

If F is not a reversible Finsler metric, then the volume $\mathrm{vol}_{\dot{g}_x}(S_xM)$ may be greater than $\alpha(n-1)$ (see [3]).

Proof. Recall that given a convex set D_xM in a vector space T_xM, the dual set D^*_xM in the dual tangent space T^*_xM is defined by

$$D^*_xM := \{\xi \in T^*_xM : \xi(y) \leq 1 \text{ for all } y \in D_xM\}.$$

By Blaschke-Santaló's inequality [12] which requires the symmetry of D_xM and D^*_xM, we have $\mathrm{vol}(D_xM) \cdot \mathrm{vol}(D^*_xM) \leq \beta(n)^2$, and equality holds if and only if D_xM is an ellipsoid.

Note that the map

$$y^i \mapsto \sum_{j=1}^n g_{ij}(x, y)y^j$$

is a diffeomorphism between D_xM and D^*_xM with Jacobian $\det\big(g_{ij}(x, y)\big)$. Therefore

$$\int\limits_{D_xM} \det\big(g_{ij}(x, y)\big)\, dy = \int\limits_{D^*_xM} 1\, d\xi = \mathrm{vol}(D^*_xM).$$

Then we have that

$$\begin{aligned}
\mathrm{vol}_{\hat{g}_x}(D_xM) &:= \int\limits_{D_xM} 1\, d\hat{g}_x = \int\limits_{D_xM} \sqrt{\det\big(g_{ij}(x,y)\big)}\, dy \\
&\le \Bigg\{ \int\limits_{D_xM} \det\big(g_{ij}(x,y)\big)\, dy \Bigg\}^{1/2} \cdot \Bigg\{ \int\limits_{D_xM} 1\, dy \Bigg\}^{1/2} \\
&= \{\mathrm{vol}(D_x^*M)\cdot \mathrm{vol}(D_xM)\}^{1/2} \\
&\le \beta(n),
\end{aligned}$$

and equality holds if and only if $\det(g_{ij}(x,y))$ is constant for y and D_xM is an ellipsoid. □

The idea of the proof is borrowed from [7], where Duran established a sharp upper bound on the volume of the Sasaki metric on the unit tangent sphere bundle. Here we are only concerned with the volume of the induced Riemannian metric on the unit tangent sphere.

The Chern connection on a Finsler manifold M is defined by the unique set of local 1-forms $\{{\omega_j}^i\}_{1\le i,j\le n}$ on $\widetilde{TM}$ such that

$$\begin{aligned}
d\omega^i &= \sum_{j=1}^n \omega^j \wedge {\omega_j}^i, \\
dg_{ij} &= \sum_{k=1}^n \left(g_{kj}{\omega_i}^k + g_{ik}{\omega_j}^k + 2A_{ijk}{\omega_n}^k \right), \text{ where } A_{ijk} = \frac{\partial g_{ij}}{\partial y^k}.
\end{aligned}$$

Define the set of local curvature forms ${\Theta_j}^i$ by

$$ {\Theta_j}^i := \sum_{k=1}^n \left(d{\omega_j}^i - {\omega_j}^k \wedge {\omega_k}^i \right). $$

Then we can write

$$ {\Theta_j}^i = \sum_{k,l=1}^n \left(\frac{1}{2} {R_j}^i{}_{kl}\,\omega^k \wedge \omega^l + {P_j}^i{}_{kl}\,\omega^k \wedge \omega^{n+l} \right). $$

Define the curvature tensor R by

$$ R(U,V)W = \sum_{i,j,k,l=1}^n u^k v^l w^j {R_j}^i{}_{kl} E_i, $$

where $U = \sum_{i=1}^n u^i E_i$, $V = \sum_{i=1}^n v^i E_i$ and $W = \sum_{i=1}^n w^i E_i$ are vectors in the pull-back bundle π^*TM of TM by $\pi : \widetilde{TM} \to M$. For a fixed $v \in T_xM$, let γ_v be the geodesic from $\gamma_v(0) = x$ with $\dot\gamma_v(0) = v$. Along γ_v, we have the *osculating Riemannian metrics*

$$ g^{\dot\gamma_v(t)} := g\big(\gamma_v(t), \dot\gamma_v(t)\big) $$

in $T_{\gamma_v(t)}M$. Define the *flag curvature* $R^{\dot{\gamma}_v(t)}(u(t)) : T_{\gamma_v(t)}M \to T_{\gamma_v(t)}M$ by

$$R^{\dot{\gamma}_v(t)}(u(t)) := R\big(U(t), V(t)\big)V(t),$$

where $U(t) = \big(\hat{\gamma}_v(t); u(t)\big), V(t) = \big(\hat{\gamma}_v(t); \gamma_v(t)\big) \in \pi^*TM$. The flag curvature is independent of connections, that is, this term appears in the second variation formula of arc length, thus is of particular interest to us. We remark that if F is Riemannian, then the flag curvature coincides with the sectional curvature.

Define a map $\psi_x : (0, \infty) \times S_xM \to M$ by $\psi_x(t, v) = \gamma_v(t)$. The following proposition proved by Shen [13, Theorem 0.1] will play a crucial role in this paper.

Proposition 1.2. *Let (M, F) be a simply connected reversible Finsler manifold with constant flag curvature one. Then for every $x \in M$, there is a unique point $x^* \in M$ with* $\mathrm{dist}(x, x^*) = \pi$, *and every geodesic issuing from x is closed with length 2π, passing through x^*. The map ψ_x restricted to $(0, \pi) \times S_xM$ is a diffeomorphism to $M \setminus \{x, x^*\}$. Furthermore, the osculating Riemannian metric $g^{\dot{\gamma}_v(t)}$ has the form*

$$(\psi_x)^* \left(g^{\dot{\gamma}_v(t)}\right) = dt^2 \oplus \sin^2 t\ \dot{g}_x.$$

Remark 1.3. In this case, the osculating Riemannian metric $g^{\dot{\gamma}_v(t)}$ on $M\setminus\{x, x^*\}$ is independent of the direction $v \in S_xM$ and dependent of the induced Riemannian metric $\dot{g}_x$ on S_xM and the distance from x. Hence the volume density

$$dg^{\dot{\gamma}_v(t)} = \sqrt{\det\left(g_{ij}^{\dot{\gamma}_v(t)}\right)}\ dx$$

is expressed by

$$(\psi_x)^* \left(dg^{\dot{\gamma}_v(t)}\right) = \sqrt{\det\left((\psi_x)^* \left(g_{ij}^{\dot{\gamma}_v(t)}\right)\right)}\, dt\, d\dot{g}_x = \sin^{n-1} t\, dt\, d\dot{g}_x,$$

and does not dependent of the direction v.

By using Lemma 1.1 and Proposition 1.2, we obtain the following lemma.

Lemma 1.4. *For M and $g^{\dot{\gamma}_v(t)}$ defined above, we have that the volume* $\mathrm{vol}_{g^{\dot{\gamma}_v(t)}}(M)$ *of M with respect to the osculating Riemannian metric $g^{\dot{\gamma}_v(t)}$ on $M \setminus \{x, x^*\}$ less than equality to $\alpha(n)$ and equality holds if and only if (T_xM, F_x) is Euclidean.*

Proof. By Proposition 1.2, we obtain

$$\begin{aligned}
\mathrm{vol}_{g^{\dot\gamma_v(t)}}(M) &:= \int\limits_{M\setminus\{x,x^*\}} 1\, dg^{\dot\gamma_v(t)} \\
&= \int\limits_{(0,\pi)\times S_xM} \sqrt{\det\Big((\psi_x)^*\,(g_{ij}^{\dot\gamma_v(t)})\Big)}\, dt\, d\dot g_x \\
&= \mathrm{vol}_{\dot g_x}(S_xM)\cdot \int\limits_0^\pi \sin^{n-1} t\, dt \\
&\le \alpha(n-1)\cdot \int\limits_0^\pi \sin^{n-1} t\, dt = \alpha(n).
\end{aligned}$$

We note that the last line is obtained from Lemma 1.1 and equality holds if and only if (T_xM, F_x) is Euclidean. □

2. Symplectic volumes. In this section we prove our main theorem.

Let $\{\frac{\partial}{\partial x^i}\}_{i=1}^n$ be a local basis for TM and $\{dx^i\}_{i=1}^n$ be its dual basis for T^*M. By the Chern connection, we obtain the decomposition

$$T^*(\widetilde{TM}) = \mathrm{span}\{dx^i\} \oplus \mathrm{span}\{\delta y^i\},$$

where δy^i is the vertical component dy^i and is given by

$$\delta y^i = dy^i + \sum_{j=1}^n N_j^i dx^j$$

for some N_i^j determined by the Chern connection. Then there is a naturally induced Sasaki-Riemannian metric h on $\widetilde{TM}$ defined by

$$h\big((x,y)\big) := \sum_{i,j=1}^n \big(g_{ij}(x,y)\ dx^i\otimes dx^j \oplus g_{ij}(x,y)\ \delta y^i \otimes \delta y^j\big)\,,$$

and the volume form dV of h on $\widetilde{TM}$ is given by

$$\begin{aligned}
dV((x,y)) &:= \sqrt{\det\big(g_{ij}(x,y)\big)}\ dx^1\wedge\cdots\wedge dx^n \cdot \sqrt{\det\big(g_{ij}(x,y)\big)}\ \delta y^1\wedge\cdots\wedge \delta y^n \\
&= \sqrt{\det\big(g_{ij}(x,y)\big)}\ dx^1\wedge\cdots\wedge dx^n \cdot \sqrt{\det\big(g_{ij}(x,y)\big)}\ dy^1\wedge\cdots\wedge dy^n \\
&= dg^y \wedge d\hat g_x.
\end{aligned}$$

Let $\omega = \sum_{i=1}^n \frac{\partial F}{\partial y^i} dx^i$ be the Hilbert 1-form on $\widetilde{TM}$. In local coordinates, we have the volume form

$$dV = \frac{1}{n!}\underbrace{d\omega\wedge\cdots\wedge d\omega}_{\text{n-times}}$$

on $\widetilde{TM}$.

There is another interpretation of this volume on tangent space. Let SM be the unit tangent bundle on M and $i : SM \to \widetilde{TM}$ the natural embedding. Let X_ω be the Reeb field of the Hilbert 1-form ω. It is uniquely determined by the conditions $\omega(X_\omega) = 1, i_{X_\omega}(d\omega) = 0$. In particular we have $L_{X_\omega}\omega = 0$ and the geodesic flow of Finsler metric, i.e., the flow with infinitesimal generator X_ω, consists of contact diffeomorphisms and the volume form $i^*(dV)$ on SM is

$$dV = \frac{1}{(n-1)!}\ \omega \wedge \underbrace{d\omega \wedge \cdots \wedge d\omega}_{\text{(n-1)-times}} = d\dot{g}_x \wedge dg^y.$$

Since $L_{X_\omega}\omega = 0$, the volume form is invariants under the geodesic flow of Finsler metric. We shall use the same notation dV for the volume forms of TM and SM, if no confusion is caused.

By using Remark 1.3 and Lemma 1.4, we obtain the following theorem.

Theorem 2.1. *Let (M, F) be an n-dimensional simply connected reversible Finsler manifold with constant flag curvature one. Then we have*

$$V(SM) \le \alpha(n-1) \cdot \alpha(n),$$

and equality holds if and only if (M, F) is a Riemannian manifold.

Proof. By Proposition 1.2, for all $x \in M$, the map $\psi_x : (0,\pi)\times S_xM \to M\backslash\{x, x^*\}$, $\psi_x(t, v) \mapsto \gamma_v(t)$ is a diffeomorphism and we have that

$$\begin{aligned} V(SM) &= V\left(\bigcup_{\gamma_v(t)\in M\backslash\{x,x^*\}} S_{\gamma_v(t)}M\right) \\ &= \int_{\left(\cup_{\gamma_v(t)\in M\backslash\{x,x^*\}} S_{\gamma_v(t)}M\right)} 1\, d\dot{g}_{\gamma_v(t)} \wedge dg^{\dot{\gamma}_v(t)} \\ &= \int_{M\backslash\{x,x^*\}} \left\{ \int_{S_{\gamma_v(t)}M} 1\, d\dot{g}_{\gamma_v(t)} \right\} dg^{\dot{\gamma}_v(t)} \\ &= \int_{M\backslash\{x,x^*\}} \left(\mathrm{vol}_{\dot{g}_{\gamma_v(t)}}\left(S_{\gamma_v(t)}M\right)\right) dg^{\dot{\gamma}_v(t)} \\ &\le \alpha(n-1)\cdot\alpha(n). \end{aligned}$$

We note that the third line is obtained from Remark 1.3 and the last line is obtained from Lemmas 1.1 and 1.4 and equality holds if and only if (M, F) is a Riemannian manifold. □

The notion of symplectic structure came up in Weinstein's work on the Blaschke conjecture in [15]. He proved that for an n-dimensional Riemannian manifold M all of whose geodesics are closed and of the same length 2π, the ratio $\mathrm{vol}(M)/\big((2\pi)^n \cdot$

$\alpha(n)\big)$ is an integer. The symplectic structure of Weinstein's proof implies that they can be extended to Finsler manifolds with little modification. Since the Riemannian relation the volume $V(SM)$ of SM with respect to the volume form dV, $V(SM) = \alpha(n-1)\cdot \mathrm{vol}(M)$ breaks down in the Finsler case, we rewrite Weinstein's result as follows.

Theorem 2.2. *Let (M,F) be an n-dimensional Finsler manifold all of whose geodesics are closed and of the same length 2π. Then the ratio*

$$i(M) = \frac{V(SM)}{V(S\mathbb{S}^n)}$$

is an integer.

Proof. Since the orbits of the geodesic spray are all periodic with 2π, the geodesic flow on the SM defines a fixed point free $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$-action, whose orbits are identified with closed geodesics of length 2π. Therefore, the orbit space $SM/\mathbb{S}^1$ may be considered as a $2(n-1)$-dimensional manifold CM of all closed geodesics of M. Let $p : SM \to CM$ be the canonical projection which sends a given unit vector to the geodesic which has this vector as initial condition. Hence the projection $p : SM \to CM$ is a principle bundle with structure group $\mathbb{S}^1$, and Álvarez Paiva [1] proved that if dw is the standard symplectic form on TM, then there is a unique symplectic form Ω on CM which satisfies the equation

$$p^*\Omega = i^*\Big(\frac{d\omega}{2\pi}\Big) = \frac{1}{2\pi}\cdot d\omega.$$

From the Fubini theorem for fibrations we get

$$\begin{aligned} V(SM) &= \int\limits_{SM} \frac{1}{(n-1)!}\,\omega\wedge(d\omega)^{n-1} \\ &= \frac{1}{(n-1)!}\int\limits_{SM} \omega\wedge p^*(2\pi\Omega)^{n-1} \\ &= \frac{(2\pi)^{n-1}}{(n-1)!}\int\limits_{x\in CM}\left(\int\limits_{p^{-1}(x)}\omega\right)\Omega^{n-1}. \end{aligned}$$

Now we set

$$j(M) := \int\limits_{CM}\Omega^{n-1}.$$

Then $j(M)$ is a topological invariant of the fibration $p : SM \to CM$. We adapt Weinstein's argument (see [15]) to see that the integer $j(M)$ is an even integer $2\cdot i(M)$. However we know $\int_{p^{-1}(x)}\omega = 2\pi$ and

$$V(SM) = \frac{(2\pi)^n}{(n-1)!}\int\limits_{CM}\Omega^{n-1} = \frac{(2\pi)^n}{(n-1)!}2\cdot i(M).$$

Since $2 \cdot (2\pi)^n/(n-1)! = \alpha(n-1) \cdot \alpha(n) = V(S\mathbb{S}^n)$, we obtain the equality as stated in the theorem. □

Under the assumption of Theorem 2.2, if M is homeomorphic to the sphere, Weinstein [15] and Yang [16] showed that the topological invariant $i(M)$ is equal to one.

Now we are ready to prove main theorem using Theorems 2.1 and 2.2.

Theorem 2.3. *If (M, F) is an n-dimensional reversible Finsler manifold with constant flag curvature one, then F is a Riemannian metric.*

Proof. We first recall that if the universal covering $\overline{M}$ of M is an n-dimensional reversible Finsler manifold with constant flag curvature one, by Proposition 1.2 we know that every geodesic is closed with same length 2π and $\overline{M}$ is a diffeomorphic to the n-dimensional standard sphere. Thus by Theorem 2.2 and the above remark, the symplectic volume of $\overline{M}$, $V(S\overline{M})$ is equal to $\alpha(n-1) \cdot \alpha(n)$. Hence by Theorem 2.1, F is a Riemannian metric. □

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